Università degli Studi di Milano

# **Propositional logic**

Sandro Zucchi

2022-23

S. Zucchi: Language and Logic - Propositional logic

# The language LP

the symbols

- ▶ An infinite set of propositional letters:  $p_1 p_2 p_3 p_4 \ldots$
- $\blacktriangleright$  The connectives:  $\land ~\lor ~\supset ~\equiv ~\sim$
- ► The parentheses: ( )
- (Convention: in the examples, we will use "p", "q", "r", "s"...as propositional letters).

# Defining a formal language



- We define a formal language LP (belonging to a family of languages called *languages of propositional logic*).
- ► To define a language, we must answer these questions:
  - 1. what are the symbols of the language (the lexicon)?
  - 2. what are the grammatical sequences of symbols of the language (the *sentences* or *well-formed formulae*)?
  - 3. what is the semantics of the language, namely what are the circumstances that make the sentences of the language true?

S. Zucchi: Language and Logic – Propositional logic

The language LP

1

#### the well-formed formulae

- (a) The propositional letters are well-formed formulae of LP (the *atomic formulae*).
  - If  $\varphi \in \psi$  are well-formed formulae of LP, then:
- (b)  $\lceil \sim \varphi \rceil$  is a well-formed formula of LP,
- (c)  $\lceil (\varphi \land \psi) \rceil$  is a well-formed formula of LP,
- (d)  $\lceil (\varphi \lor \psi) \rceil$  is a well-formed formula of LP,
- (e)  $\ulcorner(\varphi \supset \psi)\urcorner$  is a well-formed formula of LP,
- (f)  $\ulcorner(\varphi \equiv \psi) \urcorner$  is a well-formed formula of LP.
- (g) Nothing else is a well-formed formula of LP.
- (Convention: we can leave the parentheses out, when it doesn't create ambiguities).

# The language LP

### valuations

- A valuation of LP is a function v which assigns a truth value (1, true, or 0, false) to the well-formed formulae of LP and meets these conditions:
  - (a) if  $\varphi$  is a propositional letter of LP,  $\nu(\varphi) \in \{0, 1\}$ ; if  $\varphi$  and  $\psi$  are well-formed formulae of LP, then:
  - (b)  $\nu(\sim \varphi) = 1$  if  $\nu(\varphi) = 0$ , otherwise  $\nu(\sim \varphi) = 0$ ;
  - (c)  $\nu(\varphi \land \psi) = 1$  if  $\nu(\varphi) = 1$  and  $\nu(\psi) = 1$ , otherwise  $\nu(\varphi \land \psi) = 0$ ;
  - (d)  $\nu(\varphi \lor \psi) = 1$  if it is not the case that  $\nu(\varphi) = 0$  and  $\nu(\psi) = 0$ , otherwise  $\nu(\varphi \lor \psi) = 0$ ;
  - (e)  $\nu(\varphi \supset \psi) = 1$  if it is not the case that  $\nu(\varphi) = 1$  and  $\nu(\psi) = 0$ , otherwise  $\nu(\varphi \supset \psi) = 0$ ;

(f) 
$$\nu(\varphi \equiv \psi) = 1$$
 if  $\nu(\varphi) = \nu(\psi)$ , otherwise  $\nu(\varphi \equiv \psi) = 0$ 

S. Zucchi: Language and Logic - Propositional logic

# Comments on the definition of valuation

- The definition of valuation specifies the truth conditions of formulae of the form Γ~ φ¬, Γ(φ ∧ ψ)¬, Γ(φ ∨ ψ)¬, Γ(φ ⊃ ψ)¬ e Γ(φ ≡ ψ)¬. For example, Γ(~ φ)¬ is true exactly in those cases in which φ is false; Γ(φ ∧ ψ)¬ is true exactly in those cases in which φ is true; etc.
- While the truth value of non atomic formulae in a valuation depends on the value of other formulae in that valuation, the truth value of a propositional letter in a valuation does not depend on the truth value of other propositional letters in that valuation (since any arbitrary assignment of truth values to the propositional letters determines a valuation).
- (This feature of valuations is important if one tries to use LP to represent arguments formulated in English).

# The language LP

### truth tables

An alternative notation to express conditions (b)-(f) in the definition of valuation is the following:

$\begin{array}{c} \varphi \\ 1 \\ 1 \\ 0 \end{array}$	ψ 1 0 1	$\begin{array}{c} (\varphi \land \psi) \\ 1 \\ 0 \\ 0 \end{array}$	$egin{array}{c} arphi \\ arphi \\ arphi \\ arphi \\ arphi \\ arphi \end{array}$	ψ 1 0 1	$\begin{array}{c c} (\varphi \ \lor \ \psi) \\ \hline 1 \\ 1 \\ 1 \\ 1 \end{array}$	$\begin{array}{c} \varphi \\ 1 \\ 1 \\ 0 \end{array}$	ψ 1 0 1	$\begin{array}{c c} (\varphi \supset \psi) \\ \hline 1 \\ 0 \\ 1 \\ \end{array}$	
0 φ	0 ψ	$  0 \\ (\varphi \equiv \psi)$	0 φ	0	0 • <i>o</i>	0	0	1	
$\frac{\varphi}{1}$	$\frac{\gamma}{1}$	$\frac{(\varphi - \varphi)}{1}$	$\frac{\varphi}{1}$		$\frac{\varphi}{0}$				
1	0	0	0		1				
0	1	0		1					
0	0	1							
S. Zucchi: Language and Logic – Propositional logic									6

# Validity in LP

- An argument in LP consists of a set of formulae (the premises) and a formula (the conclusion).
- An argument with premises {φ<sub>1</sub>,..., φ<sub>n</sub>} and conclusion ψ is valid in LP if and only if there is no valuation that makes φ<sub>1</sub>,..., φ<sub>n</sub> true and ψ false in LP.
- If an argument is valid in LP we will say that its premises logically implicate its conclusion.
- ▶ In symbols, when an argument is valid in LP, we shall write:

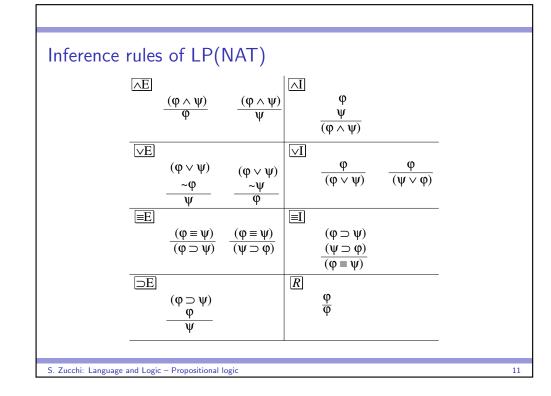
 $\{\varphi_1,\ldots,\varphi_n\}\models_{LP}\psi$ 

► A well-formed formula  $\varphi$  of LP is valid in LP ( $\models_{LP} \varphi$ ) if and only if every valuation makes it true.

# Natural deduction for propositional logic

- We will now introduce a natural deduction system for propositional logic (called LP(NAT) for brevity). Namely, we will introduce some rules for the language LP which allow us to derive a conclusion from a set of premises.
- These rules are purely syntactic, namely they make no reference to the meanings of formulae they manipulate.
- However, the *justification* of these rules is *semantic*, since the rules we introduce allow us to derive a conclusion from the premises exactly in those cases in which the premises logically implicate the conclusion.

S. Zucchi: Language and Logic - Propositional logic



# The system LP(NAT)

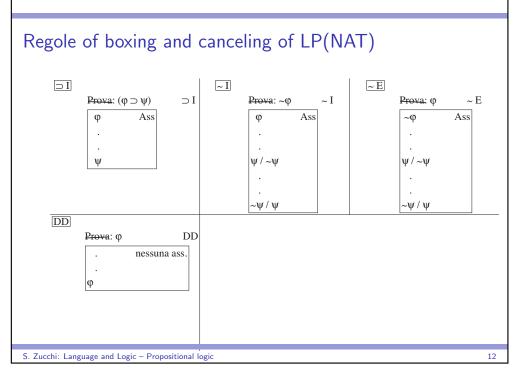
- LP(NAT) consists of two kinds of rules: rules of inference and rules of boxing and canceling.
- ► For each connective, we'll introduce two rules:
  - one that allows to prove a formula with that connective as a main connective (introduction rule),
  - one that, starting from a formula with that connective as a main connective, allows us to prove a formula that does not have that connective as a main connective (elimination rule).

10

 (The system is based on Kalish and Montague 1964, and Gettier 1984).

S. Zucchi: Language and Logic - Propositional logic

9



## Derivation in a natural deduction system

# The notion of *derivation* in a natural deduction system is now defined thus:

a derivation is a sequence of numbered lines, built according to the general rules to build derivations, where every unboxed line is a premise (a line glossed as 'P') or a canceled line 'prova' (proof).

S. Zucchi: Language and Logic - Propositional logic

## Notation

- ⊢<sub>s</sub> φ =<sub>def.</sub> There is a derivation in the system s which contains no lines annotated 'P', and □Prova:φ□ is the only unboxed line.
- Γ ⊢<sub>s</sub> φ =<sub>def</sub>. There is a derivation in a system s in which every line annotated 'P' is a member of Γ, and Γ<del>Prova</del>:φ<sup>¬</sup> is the only unboxed line which is not a premise.

# General rules to build derivations

- Rule for lines 'Prova': for every well-formed formula φ, it is aways possible to introduce a line ΓProva:φ<sup>¬</sup>
- Rule for premises: any well-formed formula  $\varphi$  may be entered as a line glossed 'P', provided no line 'Prova' has been entered.
- Rule for assumptions: any well-formed formula φ may be introduced as a line glossed 'Ass' on a line immediately following a line 'Prova'.
- Rule for inferences: any well-formed formula φ may be introduced as a line if it follows from available lines by a rule of inference, provided it is glossed with the name of the rule of inference and with the numbers of the lines from which it is inferred. (A line is available provided it is not boxed and contains no uncanceled 'Prova').
- Rule for boxing and canceling: the last uncanceled line 'Prova' may be canceled by means of a rule of boxing and canceling, provided that the line 'Prova' is glossed with the name of the rule.

S. Zucchi: Language and Logic – Propositional logic

## Completeness and correctness

- It is possible to show (although we won't do it here) that
- there is a derivation of ψ from {φ<sub>1</sub>,..., φ<sub>n</sub>} by the rules of LP(NAT) if and only if the argument that has φ<sub>1</sub>,..., φ<sub>n</sub> as premises and ψ as a conclusion is valid in LP.
- In symbols, we can write this result thus:

 $\{\varphi_1,\ldots,\varphi_n\}\vdash_{LP(NAT)}\psi$  iff  $\{\varphi_1,\ldots,\varphi_n\}\models_{LP}\psi$ .

• (As a particular case, one can show that  $\vdash_{LP(NAT)} \psi$  iff  $\models_{LP} \psi$ ).

13

14