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Propositional logic

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Defining a formal language

- We define a formal language LP (belonging to a family of languages called *languages of propositional logic*).
- To define a language, we must answer these questions:
  1. what are the symbols of the language (the lexicon)?
  2. what are the grammatical sequences of symbols of the language (the *sentences* or *well-formed formulae*)?
  3. what is the semantics of the language, namely what are the circumstances that make the sentences of the language true?

The language LP

the symbols

- An infinite set of propositional letters: $p_1 \quad p_2 \quad p_3 \quad p_4 \quad \dots$
- The connectives: $\wedge \quad \vee \quad \supset \quad \equiv \quad \sim$
- The parentheses: ( )

- (Convention: in the examples, we will use "*p*", "*q*", "*r*", "*s*"... as propositional letters).

The language LP

the well-formed formulae

(a) The propositional letters are well-formed formulae of LP (the *atomic formulae*).

If φ e ψ are well-formed formulae of LP, then:

(b) $\ulcorner \sim \varphi \urcorner$ is a well-formed formula of LP,

(c) $\ulcorner (\varphi \wedge \psi) \urcorner$ is a well-formed formula of LP,

(d) $\ulcorner (\varphi \vee \psi) \urcorner$ is a well-formed formula of LP,

(e) $\ulcorner (\varphi \supset \psi) \urcorner$ is a well-formed formula of LP,

(f) $\ulcorner (\varphi \equiv \psi) \urcorner$ is a well-formed formula of LP.

(g) Nothing else is a well-formed formula of LP.

- (Convention: we can leave the parentheses out, when it doesn't create ambiguities).

The language LP

valuations

- A *valuation* of LP is a function ν which assigns a truth value (1, true, or 0, false) to the well-formed formulae of LP and meets these conditions:
  (a) if φ is a propositional letter of LP, $\nu(\varphi) \in \{0, 1\}$;
  if φ and ψ are well-formed formulae of LP, then:
  (b) $\nu(\sim \varphi) = 1$ if $\nu(\varphi) = 0$, otherwise $\nu(\sim \varphi) = 0$;
  (c) $\nu(\varphi \wedge \psi) = 1$ if $\nu(\varphi) = 1$ and $\nu(\psi) = 1$, otherwise $\nu(\varphi \wedge \psi) = 0$;
  (d) $\nu(\varphi \vee \psi) = 1$ if it is not the case that $\nu(\varphi) = 0$ and $\nu(\psi) = 0$, otherwise $\nu(\varphi \vee \psi) = 0$;
  (e) $\nu(\varphi \supset \psi) = 1$ if it is not the case that $\nu(\varphi) = 1$ and $\nu(\psi) = 0$, otherwise $\nu(\varphi \supset \psi) = 0$;
  (f) $\nu(\varphi \equiv \psi) = 1$ if $\nu(\varphi) = \nu(\psi)$, otherwise $\nu(\varphi \equiv \psi) = 0$.

The language LP

truth tables

An alternative notation to express conditions (b)-(f) in the definition of valuation is the following:

φ	ψ	$(\varphi \wedge \psi)$
1	1	1
1	0	0
0	1	0
0	0	0

φ	ψ	$(\varphi \vee \psi)$
1	1	1
1	0	1
0	1	1
0	0	0

φ	ψ	$(\varphi \supset \psi)$
1	1	1
1	0	0
0	1	1
0	0	1

φ	ψ	$(\varphi \equiv \psi)$
1	1	1
1	0	0
0	1	0
0	0	1

φ	$\sim \varphi$
1	0
0	1

Comments on the definition of valuation

- The definition of valuation specifies the truth conditions of formulae of the form $\ulcorner\sim \varphi\urcorner$, $\ulcorner(\varphi \wedge \psi)\urcorner$, $\ulcorner(\varphi \vee \psi)\urcorner$, $\ulcorner(\varphi \supset \psi)\urcorner$ e $\ulcorner(\varphi \equiv \psi)\urcorner$. For example, $\ulcorner(\sim \varphi)\urcorner$ is true exactly in those cases in which φ is false; $\ulcorner(\varphi \wedge \psi)\urcorner$ is true exactly in those cases in which φ is true and ψ is true; etc.
- While the truth value of non atomic formulae in a valuation depends on the value of other formulae in that valuation, the truth value of a propositional letter in a valuation does not depend on the truth value of other propositional letters in that valuation (since any arbitrary assignment of truth values to the propositional letters determines a valuation).
- (This feature of valuations is important if one tries to use LP to represent arguments formulated in English).

Validity in LP

- An argument in LP consists of a set of formulae (the *premises*) and a formula (the *conclusion)*.
- An argument with premises $\{\varphi_1, \ldots, \varphi_n\}$ and conclusion ψ is *valid* in LP if and only if there is no valuation that makes $\varphi_1, \ldots, \varphi_n$ true and ψ false in LP.
- If an argument is valid in LP we will say that its premises *logically implicate* its conclusion.
- In symbols, when an argument is valid in LP, we shall write:

$$\{\varphi_1, \ldots, \varphi_n\} \models_{LP} \psi$$

- A well-formed formula φ of LP is valid in LP ($\models_{LP} \varphi$) if and only if every valuation makes it true.

Natural deduction for propositional logic

- We will now introduce a *natural deduction system* for propositional logic (called LP(NAT) for brevity). Namely, we will introduce some rules for the language LP which allow us to *derive* a conclusion from a set of premises.
- These rules are purely *syntactic*, namely they make no reference to the meanings of formulae they manipulate.
- However, the *justification* of these rules is *semantic*, since the rules we introduce allow us to derive a conclusion from the premises exactly in those cases in which the premises logically implicate the conclusion.

The system LP(NAT)

- LP(NAT) consists of two kinds of rules: rules of inference and rules of boxing and canceling.
- For each connective, we'll introduce two rules:
  - one that allows to prove a formula with that connective as a main connective (introduction rule),
  - one that, starting from a formula with that connective as a main connective, allows us to prove a formula that does not have that connective as a main connective (elimination rule).
- (The system is based on Kalish and Montague 1964, and Gettier 1984).

Inference rules of LP(NAT)

∧E

$$\frac{(\varphi \wedge \psi)}{\varphi} \qquad \frac{(\varphi \wedge \psi)}{\psi}$$

∧I

$$\frac{\begin{array}{c}\varphi \\ \psi\end{array}}{(\varphi \wedge \psi)}$$

∨E

$$\frac{\begin{array}{c}(\varphi \vee \psi) \\ {\sim}\varphi\end{array}}{\psi} \qquad \frac{\begin{array}{c}(\varphi \vee \psi) \\ {\sim}\psi\end{array}}{\varphi}$$

∨I

$$\frac{\varphi}{(\varphi \vee \psi)} \qquad \frac{\varphi}{(\psi \vee \varphi)}$$

≡E

$$\frac{(\varphi \equiv \psi)}{(\varphi \supset \psi)} \qquad \frac{(\varphi \equiv \psi)}{(\psi \supset \varphi)}$$

≡I

$$\frac{\begin{array}{c}(\varphi \supset \psi) \\ (\psi \supset \varphi)\end{array}}{(\varphi \equiv \psi)}$$

⊃E

$$\frac{\begin{array}{c}(\varphi \supset \psi) \\ \varphi\end{array}}{\psi}$$

R

$$\frac{\varphi}{\varphi}$$

Regole of boxing and canceling of LP(NAT)

⊃I

~~Prova~~: $(\varphi \supset \psi)$ ⊃ I

φ	Ass
.	
.	
ψ	

~I

~~Prova~~: $\sim\varphi$ ~ I

φ	Ass
.	
.	
ψ / $\sim\psi$	
.	
.	
$\sim\psi$ / ψ	

~E

~~Prova~~: φ ~ E

$\sim\varphi$	Ass
.	
.	
ψ / $\sim\psi$	
.	
.	
$\sim\psi$ / ψ	

DD

~~Prova~~: φ DD

.	nessuna ass.
.	
φ	

Derivation in a natural deduction system

The notion of *derivation* in a natural deduction system is now defined thus:

- a derivation is a sequence of numbered lines, built according to the *general rules to build derivations*, where every unboxed line is a premise (a line glossed as 'P') or a canceled line 'prova' (proof).

General rules to build derivations

- *Rule for lines 'Prova':* for every well-formed formula φ, it is aways possible to introduce a line ⌜Prova:φ⌝
- *Rule for premises:* any well-formed formula φ may be entered as a line glossed '*P*', provided no line 'Prova' has been entered.
- *Rule for assumptions:* any well-formed formula φ may be introduced as a line glossed '*Ass*' on a line *immediately* following a line 'Prova'.
- *Rule for inferences:* any well-formed formula φ may be introduced as a line if it follows from *available* lines by a rule of inference, provided it is glossed with the name of the rule of inference and with the numbers of the lines from which it is inferred. (A line is *available* provided it is not boxed and contains no uncanceled 'Prova').
- *Rule for boxing and canceling:* the *last* uncanceled line 'Prova' may be canceled by means of a rule of boxing and canceling, provided that the line 'Prova' is glossed with the name of the rule.

Notation

- $\vdash_s \varphi =_{def.}$ There is a derivation in the system *s* which contains no lines annotated '*P*', and ⌜~~Prova~~:φ⌝ is the only unboxed line.
- $\Gamma \vdash_s \varphi =_{def.}$ There is a derivation in a system *s* in which every line annotated '*P*' is a member of Γ, and ⌜~~Prova~~:φ⌝ is the only unboxed line which is not a premise.

Completeness and correctness

It is possible to show (although we won't do it here) that

- there is a derivation of ψ from $\{\varphi_1, \ldots, \varphi_n\}$ by the rules of LP(NAT) if and only if the argument that has $\varphi_1, \ldots, \varphi_n$ as premises and ψ as a conclusion is valid in LP.
- In symbols, we can write this result thus:

$$\{\varphi_1, \ldots, \varphi_n\} \vdash_{LP(NAT)} \psi \text{ iff } \{\varphi_1, \ldots, \varphi_n\} \models_{LP} \psi.$$

- (As a particular case, one can show that $\vdash_{LP(NAT)} \psi$ iff $\models_{LP} \psi$).